Bahan kuliah

FISIKA KOMPUTASI

COMPUTATIONAL PHYSICS

SISTEM PERSAMAAN ALJABAR LINEAR (*LINEAR ALGEBRAIC EQUATIONS*)

Referensi:

Numerical Methods for Engineers

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LINEAR ALGEBRAIC EQUATIONS

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \qquad\qquad\qquad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

Matrix Notation

Matrices with row dimension $n = 1$, such as

$$[B] = [b_1 \quad b_2 \quad \cdots \quad b_m]$$

are called *row vectors*.

Matrices with column dimension $m = 1$, such as

$$[C] = \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix}$$

are referred to as *column vectors*.

Matrix Notation

Matrices where $n = m$ are called *square matrices.*

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Matrix Operating Rules

Addition of two matrices, say, [A] and [B], is accomplished by adding corresponding terms in each matrix. The elements of the resulting matrix [C] are computed.

$$c_{ij} = a_{ij} + b_{ij}$$

for i = 1, 2, . . . , n and j = 1, 2, . . . , m.

Similarly, the subtraction of two matrices, say, [E] minus [F], is obtained by subtracting corresponding terms, as in

$$d_{ij} = e_{ij} - f_{ij}$$

for i = 1, 2, . . . , n and j = 1, 2, . . . , m.

Matrix Operating Rules

The multiplication of a matrix [A] by a scalar g is obtained by multiplying every element of [A] by g, as in

$$[D] = g[A] = \begin{bmatrix} ga_{11} & ga_{12} & \cdots & ga_{1m} \\ ga_{21} & ga_{22} & \cdots & ga_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ ga_{n1} & ga_{n2} & \cdots & ga_{nm} \end{bmatrix}$$

The product of two matrices is represented as [C] =[A][B], where the elements of [C] are defined as:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

NAIVE GAUSS ELIMINATION

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \qquad (9.12a)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \qquad (9.12b)$$

$$\vdots \qquad \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \qquad (9.12c)$$

To do this, multiply Eq. (9.12a) by a_{21}/a_{11} to give

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \cdots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Now, this equation can be subtracted from Eq. (9.12*b*) to give

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \cdots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

NAIVE GAUSS ELIMINATION

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & b_1 \\ a_{21} & a_{22} & a_{23} & \vdots & b_2 \\ a_{31} & a_{32} & a_{33} & \vdots & b_3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & b_1 \\ & a'_{22} & a'_{23} & \vdots & b'_2 \\ & & a''_{33} & \vdots & b''_3 \end{bmatrix}$$

Forward elimination

$$\Downarrow$$

$$x_3 = b''_3 / a''_{33}$$

$$x_2 = (b'_2 - a'_{23}x_3)/a'_{22}$$

$$x_1 = (b_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$

Back substitution

Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}$$

for $i = n-1, n-2, \ldots, 1$

NAIVE GAUSS ELIMINATION

Pseudocode to perform (*a*) *forward* elimination and (*b*) *back* substitution.

(*a*)

```
DOFOR k = 1, n − 1
 DOFOR i = k + 1, n
   factor = a_i,k / a_k,k
   DOFOR j = k + 1 to n
    a_i,j = a_i,j − factor · a_k,j
   END DO
   b_i = b_i − factor · b_k
 END DO
END DO
```

(*b*)

```
x_n = b_n / a_n,n
DOFOR i = n − 1, 1, −1
  sum = b_i
  DOFOR j = i + 1, n
   sum = sum − a_i,j · x_j
  END DO
  x_i = sum / a_i,i
END DO
```

```pascal
Program Eliminasi_Gauss;
Uses WinCrt;
Var i,j,k,n  : Integer;
    x,c      : Array[1..10] Of Real;
    a        : Array[1..10,1..10] Of Real;
    Jumlah,faktor   : Real;

Procedure Gauss;
Begin
  For k:=1 To n-1 Do
    For i:=k+1 To n Do
     Begin
      faktor:=a[i,k]/a[k,k];
      For j:=k+1 To n Do
        a[i,j]:=a[i,j]-faktor*a[k,j];
      c[i]:=c[i]-faktor*c[k];
     End;
  x[n]:=c[n]/a[n,n];
  For i:=n-1 Downto 1 Do
   Begin
    jumlah:=0;
    For j:=i+1 To n Do
     jumlah:=jumlah+a[i,j]*x[j];
    x[i]:=(c[i]-jumlah)/a[i,i];
   End;
End;

Begin
 ClrScr;
 n:=3;
 a[1,1]:=3;      a[1,2]:=-0.1;    a[1,3]:=-0.2;    c[1]:=7.85;
 a[2,1]:=0.1;    a[2,2]:=7;       a[2,3]:=-0.3;    c[2]:=-19.3;
 a[3,1]:=0.3;    a[3,2]:=-0.2;    a[3,3]:=10;      c[3]:=71.4;

 Gauss; {memanggil procedure Gauss}

 For i:=1 to n DO
   Writeln('X[',i,'] = ',x[i]:6:4);
 Readln;
End.
```

EXAMPLE 9.5

Naive Gauss Elimination

Problem Statement. Use Gauss elimination to solve

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \tag{E9.5.1}$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \tag{E9.5.2}$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \tag{E9.5.3}$$

Carry six significant figures during the computation.

Solution. The first part of the procedure is forward elimination. Multiply Eq. (E9.5.1) by $(0.1)/3$ and subtract the result from Eq. (E9.5.2) to give

$$7.00333x_2 - 0.293333x_3 = -19.5617$$

Then multiply Eq. (E9.5.1) by $(0.3)/3$ and subtract it from Eq. (E9.5.3) to eliminate x_1. After these operations, the set of equations is

$$3x_1 \qquad - 0.1x_2 \qquad - 0.2x_3 = 7.85 \tag{E9.5.4}$$
$$7.00333x_2 - 0.293333x_3 = -19.5617 \tag{E9.5.5}$$
$$-0.190000x_2 + 10.0200x_3 = 70.6150 \tag{E9.5.6}$$

EXAMPLE 9.5

To complete the forward elimination, x_2 must be removed from Eq. (E9.5.6). To accomplish this, multiply Eq. (E9.5.5) by $-0.190000/7.00333$ and subtract the result from Eq. (E9.5.6). This eliminates x_2 from the third equation and reduces the system to an upper triangular form, as in

$$3x_1 \quad - 0.1x_2 \quad - 0.2x_3 = 7.85 \tag{E9.5.7}$$

$$7.00333x_2 - 0.293333x_3 = -19.5617 \tag{E9.5.8}$$

$$10.0120x_3 = 70.0843 \tag{E9.5.9}$$

We can now solve these equations by back substitution. First, Eq. (E9.5.9) can be solved for

$$x_3 = \frac{70.0843}{10.0120} = 7.0000 \tag{E9.5.10}$$

This result can be back-substituted into Eq. (E9.5.8):

$$7.00333x_2 - 0.293333(7.0000) = -19.5617$$

which can be solved for

$$x_2 = \frac{-19.5617 + 0.293333(7.0000)}{7.00333} = -2.50000 \tag{E9.5.11}$$

EXAMPLE 9.5

Finally, Eqs. (E9.5.10) and (E9.5.11) can be substituted into Eq. (E9.5.4):

$$3x_1 - 0.1(-2.50000) - 0.2(7.0000) = 7.85$$

which can be solved for

$$x_1 = \frac{7.85 + 0.1(-2.50000) + 0.2(7.0000)}{3} = 3.00000$$

The results are identical to the exact solution of $x_1 = 3$, $x_2 = -2.5$, and $x_3 = 7$. This can be verified by substituting the results into the original equation set

$$3(3) - 0.1(-2.5) - 0.2(7) = 7.85$$

$$0.1(3) + 7(-2.5) - 0.3(7) = -19.3$$

$$0.3(3) - 0.2(-2.5) + 10(7) = 71.4$$