Archimedes' law and potential energy: modelling and simulation with a spreadsheet

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Archimedes' law and potential energy: modelling and simulation with a spreadsheet

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This paper deals with some basic aspects of Archimedes' law. The focus is on potential energy relationships, so as to elucidate some common misunderstandings and to facilitate making bridges with other physical topics. A spreadsheet model is used as a simulation tool.

In a gravitational field, when a solid body, partially or totally immersed in a fluid, is left with no other external forces, the body will sink and the fluid will rise, or the body will rise and the fluid will sink, as long as the decrease in the potential energy of the mass that is sinking exceeds the increase in the potential energy of the mass that is rising; a situation of static equilibrium will be reached if and when a situation of *minimal total potential energy* is reached†.

Let us consider figure 1. In a situation such as in figure 1(*a*):

- When the block moves a distance $\mathrm{d}z$: its potential energy change is $\mathrm{d}Ep_\mathrm{B} = A_\mathrm{B} h_\mathrm{B} \rho_\mathrm{B} g\,\mathrm{d}z$; the corresponding change in the fluid is $\mathrm{d}Ep_\mathrm{F} = -A_\mathrm{B}\,\mathrm{d}z\,\rho_\mathrm{F} g h$; and the total variation of the potential energy is $\mathrm{d}Ep_\mathrm{TOTAL} = A_\mathrm{B} g(h_\mathrm{B}\rho_\mathrm{B} - h\rho_\mathrm{F})\mathrm{d}z$.

† The body will go below and above the static equilibrium position, in a damped oscillatory movement, until static equilibrium is reached, but in this paper the phenomena will be considered as quasi-hydrostatic.

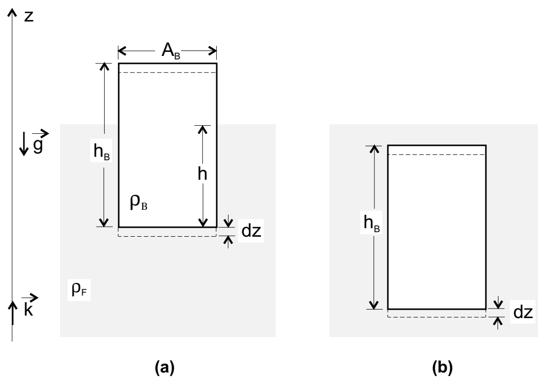

Figure 1. A_B: area of block's circular cross section; ρ_F, ρ_B: densities (fluid and block, respectively); $\boldsymbol{g}$: gravitational field. As the block sinks by $|\mathrm{d}z|$, a volume $\Delta V = A_\mathrm{B}|\mathrm{d}z|$ of fluid must rise. (*a*) ΔV rises a distance h, which will increase at the next step. (*b*) The block is totally immersed; ΔV rises a distance h_B, which will be the same at the next step.

- Thus, using the notation $\boldsymbol{F} = F_z\boldsymbol{k}$, the total force on the block is $F_z = -\mathrm{d}Ep_\mathrm{TOTAL}/\mathrm{d}z = A_\mathrm{B} g(-h_\mathrm{B}\rho_\mathrm{B} + h\rho_\mathrm{F})$. This shows that F_z has two contributions: the (downwards) weight of the block $(-A_\mathrm{B} g h_\mathrm{B}\rho_\mathrm{B})$ and the (upwards) buoyant force $(+A_\mathrm{B} g h\rho_\mathrm{F})$.
- At equilibrium, $F_z = A_\mathrm{B} g(-h_\mathrm{B}\rho_\mathrm{B} + h\rho_\mathrm{F}) = 0$ (the magnitudes of the buoyant force and of the block's weight are equal); therefore $h = (\rho_\mathrm{B}/\rho_\mathrm{F})h_\mathrm{B}$.

In a situation such as in figure 1(*b*):

- When the block moves a distance dz: its potential energy change is $dEp_B = A_B h_B \rho_B g\,dz$; the corresponding change in the fluid is $dEp_F = -A_B\,dz\,\rho_F g h_B$; and the total variation of the potential energy is $dEp_{TOTAL} = A_B g\,dz\,(h_B\rho_B - h_B\rho_F)$.
- Thus, the conclusions stated above for F_z still hold, with h_B instead of h in the expression for the buoyant force.
- At equilibrium, $F_z = -dEp_{TOTAL}/dz = A_B g(-h_B\rho_B + h_B\rho_F) = 0$; therefore $\rho_B = \rho_F$.

As is well known from elementary mathematical analysis, $dEp_{TOTAL}/dz = 0$ is a necessary condition for the occurrence of a *minimum* of Ep_{TOTAL}, the other one being $d^2Ep_{TOTAL}/dz^2 \geq 0$. In this paper, these conditions will be taken into account by a numerical approach, as follows.

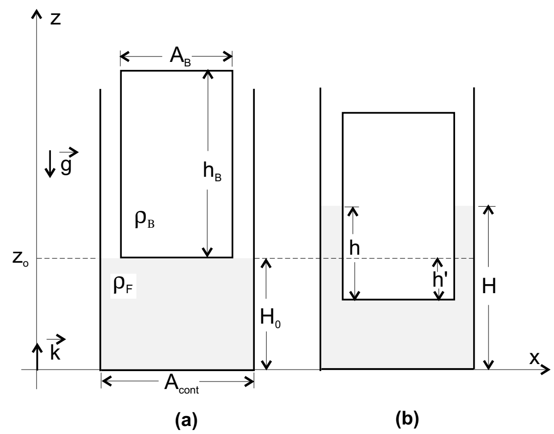

Figure 2. A_{cont}: area of container's circular cross section. (*a*) Initial situation. (*b*) An intermediate one.

Modelling

The physical interpretation given above suggests a very simple way of *step by step* calculation to find the equilibrium position (if it exists) of a body in a fluid, using a computer spreadsheet. Figure 2 is taken as a reference.

Table 1 displays part of a Microsoft Excel worksheet, for a given set of input values. Together with the displayed formulae in table 2, the following comments should be sufficient for an adequate understanding of the spreadsheet modelling†:

- At cells C8 and H8, the initial potential energy of the block and that of the fluid are computed, using their centre of mass positions. The initial situation is the one in figure 2(*a*). The x axis ($z = 0$) has been chosen as the origin (zero) for potential energies.
- At cell C10, the step displacement Δz_B of the block is specified. For *each* Δz_B:
  - ∗ The potential energy of the block changes by a fixed amount: ΔEp_B, as computed at cell C12.
  - ∗ A quantity of fluid is displaced: ΔWeight_F, as computed at cell H10.
  - ∗ The level of the fluid changes by a certain amount, ΔH. This is computed at cell H12, by expressing the conservation of the displaced volume: $-\Delta z A_B = (A_{cont} - A_B)\Delta H$.
  - ∗ The potential energy of the fluid changes by a variable amount, as computed at column F (lines 19 and below). The change that occurs when the block becomes completely immersed (cf figure 3) is taken into account by the Excel function 'IF': *if the immersed part of the block* $\leq$ *the height of block, then the potential energy variation of the fluid equals* '$\Delta \text{Weight}_F \times$ *the raise of this portion of fluid*', *else, the potential energy variation of the fluid equals* '$\Delta \text{Weight}_F \times$ *the height of the block*'.
- By copying the formulae written at line 19 and pasting them, we obtain all the formulae we need in line 20 and below, with one single operation. The symbol '$' makes the references 'absolute', so that they are not changed when formulae are copied.
- The level of the fluid after a certain number of steps can be computed as $H = H_0 + hA_B/A_{cont}$ (not displayed in table 1 for reasons of space). This expresses the conservation of the volume of fluid: $A_{cont}H_0$ (figure 2(*a*)) must equal $A_{cont}H - A_B h$ (figure 2(*b*)).

Standard Excel charts are linked to this worksheet model.

† I will be very pleased to offer a copy of the Excel file to anyone who asks for it.

Table 1. An example of an Excel worksheet. Values are introduced in the shaded cells. For printing purposes, the number of digits has been adjusted, lines 25–183 have been hidden and lines 187 and below have been omitted (the worksheet uses about 300 lines).

	B	C	D	E	F	G	H	I
1			g	9.8000	m/s^2			
3	A_B	1.800	m^2			A_{cont}	2.000	m^2
4	h_B	1.000	m			H_0	4.000	m
6	ρ_B	500.000	Kg/m^3			ρ_F	1000.000	Kg/m^3
8	In. Pot. En. $_B$	39690.00	J			In. Pot. En. $_F$	156800.00	J
10	Δz_B	-0.00030	m			Δ Weight $_F$	5.292	N
12	ΔEp_B	-2.646	J			Δ H	0.003	m
15	Block			Fluid			Block + Fluid	
16–18	Displacement h' (m)	Immersed part h (m)	Pot. En. (J)	Raise of displaced volume (m)	Pot. En. Variation (J)	Pot. En. (J)	Pot. En. Variation (J)	Pot. En. (J)
19	0.00	0.00	39687.35	0.002	0.01	156800.01	-2.64	196487.36
20	0.00	0.01	39684.71	0.005	0.02	156800.03	-2.62	196484.74
21	0.00	0.01	39682.06	0.008	0.04	156800.07	-2.61	196482.13
22	0.00	0.01	39679.42	0.011	0.06	156800.13	-2.59	196479.54
23	0.00	0.02	39676.77	0.014	0.07	156800.20	-2.57	196476.97
24				. . .				
184	0.05	0.50	39250.76	0.497	2.63	157018.74	-0.02	196269.50
185	0.05	0.50	39248.12	0.500	2.64	157021.38	0.00	196269.50
186	0.05	0.50	39245.47	0.503	2.66	157024.04	0.01	196269.51

Table 2. The same as table 1, but in another display format, i.e. displaying formulae instead of values. For printing purposes, some formulae are dislplayed in rounded rectangular text boxes.

	B	C	D	E	F	G	H	I
1			g	9.8	m/s^2			
3	A_B	1.8	m^2			A_{cont}	2	m^2
4	h_B	1	m			H_0	4	m
6	ρ_B	500	Kg/m^3	=H3*H4*H6*E1*H4/2		ρ_F	1000	Kg/m^3
8	In. Pot. En. $_B$	←	=C3*C4*C6*E1*(H4+C4/2)			In. Pot. En.$_F$	→	J
10	Δz_B	-0.0003	m	=-C3*C10*H6*E1		Δ Weight$_F$	→	N
12	ΔEp_B	←	=C3*C4*C6*E1*C10	=-C3/(H3-C3)*C10		Δ H	→	m
15	Block			Fluid			Block + Fluid	
16–18	Displac.[t] h' (m)	Immersed part h (m)	Pot. En. (J)	Raise of displaced volume (m)	Pot. En. Variation (J)	Pot. En. (J)	Pot. En. Variation (J)	Pot. En. (J)
19	=-$C$10	=-$C$10+$H$12	=$C$8+$C$12	=-$C$10/2+$H$12/2	=IF(C19<=$C	=$H$8+F19	=$C$12+F19	=D19+G19
20	=B19-$C$10	↑	=D19+$C$12	=E19-$C$10+$H$12	↑	=G19+F20	=$C$12+F20	=D20+G20
21								
22		=C19-$C$10+$H$12		=IF(C20<=$C$4,$H$10*E20,$H$10*$C$4)				

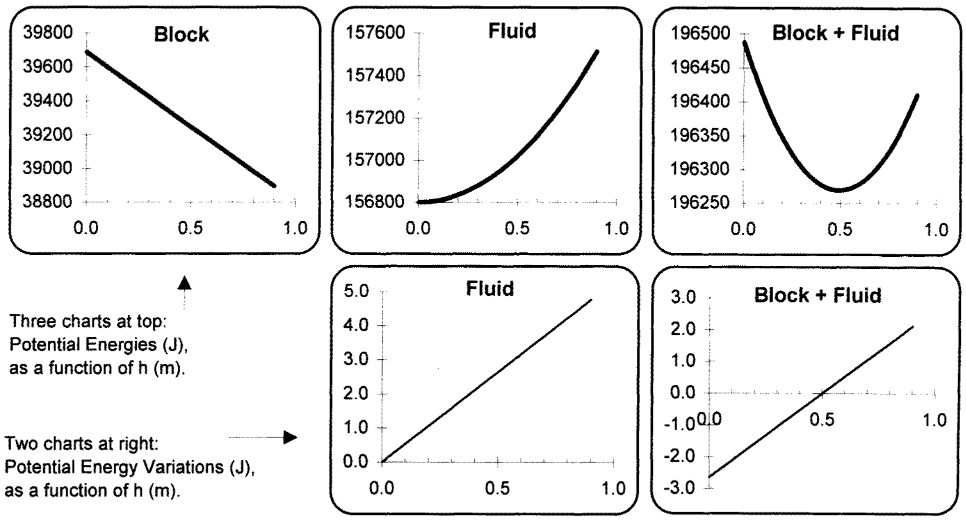

Figure 3. A set of charts that correspond to values displayed in table 1, where $\rho_B < \rho_F$.

Simulation

With the spreadsheet model, we can choose as many parameter values as desired. The numerical and graphical displays are virtually instantaneous. This allows us to bring out salient physical points to be discussed in class, as illustrated next with a few examples. We begin with the values displayed in table 1. This is a case where A_B and A_{cont} are of a same order of magnitude, and $\rho_B < \rho_F$. As we can see in table 1 and in figure 3:

- As the block goes down, its potential energy decreases and that of the fluid increases. The total potential energy decreases while $h < 0.5$ m, and then it increases. Thus, the equilibrium position (the one with a minimum of total potential energy) occurs at $h = 0.5$ m. At this position the block is thus *half* immersed, as expected from the fact that the density of the block is *half* the density of the fluid.
- At the equilibrium position, the level of the fluid is at $H = H_0 + hA_B/A_{cont} = 4 + 0.5 \times 1.8/2 = 4.45$ m. A substantial change (0.45 m) occurred, because the areas of cross section of the block (1.8 m) and of the fluid (2.0 m) have the same order of magnitude.
- That $h = 0.5$ m does *not* mean that the block went down 0.5 m. It went down only a tenth of that: as shown in table 1, line 186, h' (the displacement of the block, relative to the reference frame, between the initial and the equilibrium positions) is 0.05 m.
- The two items above allow an interpretation, in terms of energy, of the fact that 'a solid with a lower specific weight than water, even if its weight is 1000 pounds, can float in 10 pounds of water', as said Galilee, quoted by Bonera *et al* (1995).

Let us now consider the case where $A_B \gg A_{cont}$ (so as to simulate 'something like a boat on an ocean'). Let us consider, for instance, the values $A_B = 10^{-3}$ m^2 and $A_{cont} = 10$ m^2 ($A_B/A_{cont} = 10^{-4}$). This requires an increase of Δz_B, so that the equilibrium condition can be displayed within our worksheet range. A value $\Delta z_B = -0.03$ m is a good one. With these values, and maintaining all the other inputs shown in table 1, we obtain:

- At the equilibrium position we have $h = h' = 0.5$ m. Thus, the block will be half immersed, as in the previous case, but now it had to go 'all the way down'. For practical purposes, the level of the fluid did not rise. This can be confirmed: $H = H_0 + hA_B/A_{cont} = 4 + 0.5 \times 10^{-3}/10 = 4.000\,05 \simeq 4$ m.
- The charts (not displayed for this situation) will have the same general aspect as before (figure 3). Thus, if we put the block above the equilibrium position it will sink, and if we put it below that position it will rise. This also

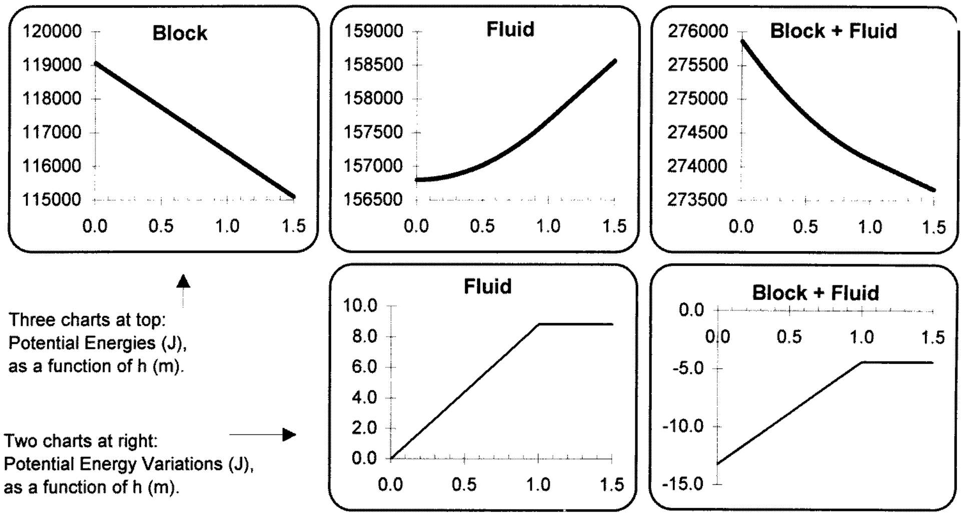

Figure 4. Charts in a case where $\rho_B > \rho_F$. Taking table 1 as a reference, the values $\rho_B = 1500$ kg m^{-3} and $\Delta z_B = -0.0005$ m have been entered, with no other changes to input values.

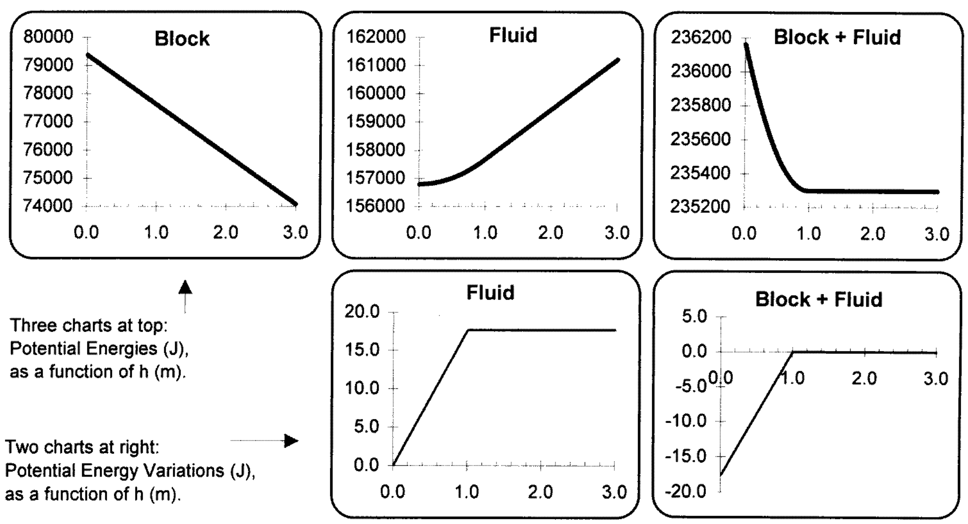

Figure 5. Charts in a case where $\rho_B = \rho_F$. Taking table 1 as a reference, the values $\rho_B = 1000$ kg m^{-3} and $\Delta z_B = -0.001$ m have been entered, with no other changes to input values.

allows an interpretation, in terms of energy, of the diving and the surfacing of a submarine, as well as of the ascent of a hot-air balloon in the atmosphere.

We return to the values in table 1, to see the effect of changing the ratio ρ_B/ρ_F. Let us consider the case where $\rho_B > \rho_F$. See figure 4:

- There will be no situation of hydrostatic equilibrium. Of course, an equilibrium position will eventually be attained, with both the fluid and the block at rest. This is when the block reaches the bottom of the container, but then there will be a *new* interaction, not taken into account in the hydrostatic physical processes.
- The rate of change of the potential energy of the fluid—and, hence, that of the total potential energy—changes at $h = 1$ m, that is to say, when the block becomes totally immersed, as already discussed in relation to figure 1.

Now, let us consider the case where $\rho_B = \rho_F$. See figure 5:

- For $h > 1$ m, the total potential energy will be constant, and the variation of the potential energy will be zero.
- This situation corresponds to a case of indifferent equilibrium (if the block is totally immersed, then it will be at equilibrium at *any* position).

As a last illustration, we consider the effect of changing the magnitude of $\boldsymbol{g}$ (charts not displayed here). With $g \neq 0$, the charts will remain unchanged, except for a matter of scale. In particular, the equilibrium positions will be the same, whether the simulations are reported to the Earth or to the Moon. Obviously, this is because both the block's and the fluid's potential energies are proportional to g. With $g = 0$, there is a symmetry breaking: all the potential energies are zero, there is no longer a difference between 'up' and 'down', and Archimedes' law is no longer valid. The same happens in a situation of weightlessness, as in the case of a satellite in free fall (Tarasov and Tarasova 1973).

Discussion and conclusion

The energy approach is powerful and flexible. Namely, it makes bridges to teaching and learning about many other physical systems, devices and situations†. For instance, essentially the same considerations apply to the study of *levers*: left alone in a gravitational field, the system (lever + weights) 'looks for' a minimum of potential energy; a given situation is one of indifferent equilibrium if the increase in the potential energy of one side of the lever causes the other side to decrease its potential energy by the same amount; the displacements that matter are those along the line of $\boldsymbol{g}$; and we cannot look only to the weights or to the displacements, but to both (and to their product, that is to say, to work, or energy). Thus, in some aspects, the study of Archimedes' law is the same as the study of levers; and, in virtually all aspects, it is the same as the study of *hydraulic presses* (which are, essentially, '*hydraulic levers*'). It should be remembered that the famous exclamation '*Eureka!*' (related to buoyancy) and the equally famous phrase '*Give me a fixed pivot point and I will raise the world!*' (related to levers) are both attributed to the same author: Archimedes, of course. This coincidence, even if it is legendary, has nowadays a physical meaning of an outstanding didactical importance.

Among many other tools, the spreadsheet can help teachers and students in the study of Archimedes' law, namely by energy considerations. Teachers can follow lines similar to the ones presented here, or they can take them as a reference to construct simpler models and simulations to use in class that do not require the use of a computer‡. Contextual characteristics and particular methodological aspects are not the aim of this paper. The central point that I want to share is this: I always felt more confident, and the teaching and learning processes always proved to be more effective, when I had in mind all the key conceptual aspects described in this paper—even where they were not fully explicitly taught to the students.

† For an application of related ideas in the scope of electrical circuits see (Silva 1995).

Acknowledgments

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Tarasov L and Tarasova A 1973 Is Archimedes' principle valid in a spaceship? *Questions and Problems in School Physics* (Moscow: MIR)

‡ On a single sheet of paper, students can construct a table with just a dozen lines and four columns, where they compute h' and the potential energies of the block, of the fluid, and the total. This is what is needed to find the equilibrium position by energy considerations. In my experience, this takes less than two hours of class time. Of course, this does not replace all the simulation facilities of the spreadsheet model, which I also use with my students.