

A Simple Classroom Simulation of Heat Energy Diffusing Through a Metal Bar

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We present an iterative procedure that does not rely on calculus to model heat flow through a uniform bar of metal and thus avoids the use of the partial differential equation typically needed to describe heat diffusion. The procedure is based on first principles and can be done with students at the blackboard. It results in a plot that illustrates several principles of thermodynamics and can be easily implemented with a spreadsheet program on the computer.

The diffusion of heat through a material by conduction can be accurately described by a diffusion equation,¹ which is beyond the scope of this paper and not used here. In the approach used here, we introduce only those equations appropriate to a first-year physics curriculum and then obtain an iterative, finite-element, step-by-step solution that simulates heat conduction and that first-year physics students should be able to carry out.

Two equations needed: the specific heat equation,² which specifies the temperature change when heat is transferred,

$$\Delta T = \frac{Q_{\text{net}}}{mc} \quad (1)$$

and the Fourier heat conduction equation,²

$$Q = \frac{(T_2 - T_1)Akt}{L}, \quad (2)$$

where T is the temperature, Q_{net} is the heat into a body minus the heat out, m is the body's mass, c is the specific heat of the substance, A is the body's cross-sectional area through which the heat flows, k is the thermal conductivity of the substance, t is the time during which the heat flows, and L is the body's length along which the heat flows. $(T_2 - T_1)/L$ is the temperature gradient in the direction of heat flow.

Assumptions to Aid the Numerical Solution

Consider the heat flow between adjacent cells within a uniform bar such as that shown in Fig. 1. We will make the following assumptions to simplify matters:

- The bar is of square cross section and each cell is a cube that occupies the entire cross section of the bar.
- We enclose the bar in a coat of perfect insulation, leaving only a hole at one end that can be quickly closed by a plug of the same material.

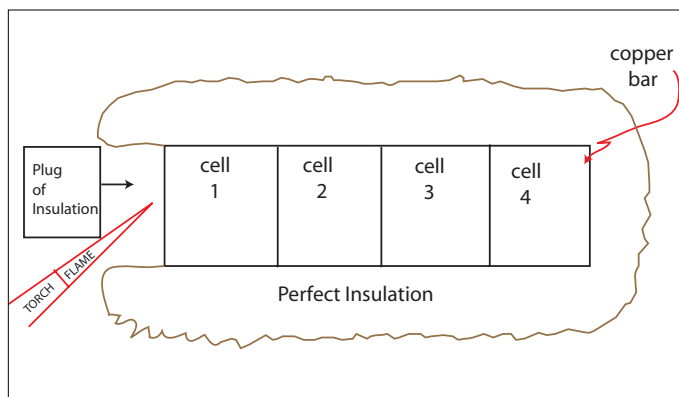


Fig. 1. The initial arrangement.

- We apply the flame of an ideal blowtorch through this hole to heat the first cell with an accurately measured amount of energy.
- We instantly replace the plug of insulation. Our first cell is thus heated to a temperature defined by Eq. (1).

The Algorithm

Our procedure is shown in the following list, which we call Table I.

Table I.

To start off, we apply a known amount of thermal energy to cell 1 as described above. After this, the algorithm is:

- Determine the internal energy of each cell in the bar.
- Determine the temperature of each cell using Eq. (1).
- Selecting a small time interval t , use Eq. (2) to find the heat flow over the cell boundaries.
- Since we know the past energy content of each cell, and the energy transfers over each cell boundary, we can then return to step (A) to calculate the energy content in each cell.

This procedure will yield the same results, albeit in quantized form, as Eqs. (1) and (2).

Now we must make some assumptions to enable our step-by-step solution. First, we assume that the energy in a cell is uniformly distributed over the volume of the cell. This is reasonable because the bar could be made of any thickness, and we know that heat travels by conduction through both thick and thin bars in a similar manner. During the conduction process the energy effectively travels a distance L , the length of a cell, when traveling to an adjacent cell.

Next, since we propose to look at the heat contained in each cell at specific times, we make the assumption that nothing unusual will happen to the heat flow between these observations, that is, that the energy flow will proceed in a predictable pattern, without reversals, peaks, or dips. This is reasonable because we are able to make the elapsed time between these intervals as short as we like.

As a corollary to this last assumption, we assume that the flow of heat across all boundaries is constant during the time interval under examination. This means that during the first time interval after we heat cell 1, we can assume that the only heat flow anywhere within the bar is across the boundary between cell 1 and cell 2, with none between cell 2 and cell 3 and none between cell 3 and cell 4. This assumption will produce a certain amount of quantization error but should not interfere with either the concepts presented here or the accuracy of the simulation.

Example Using Hand Calculations Over Four Iterations

For our example we have chosen to run our simulation on a bar of copper instead of normalizing the various constants to more simple but unrealistic values. Copper has a thermal conductivity³ of 394 W/m·K, and its specific heat is 383 J/kg·K. Its density is 8960 kg/m³.

We found it convenient to make each of our cells 5 cm, or 0.05 m, on a side. The mass of each cube is therefore 1.12 kg. For further convenience, we assume that we have refrigerated our bar to a uniform 0°C. Our ideal insulation will maintain that temperature until we apply the heat to cell 1. We have chosen to divide our example bar into four cells because this fits conveniently on a common blackboard and is not impractical to work by hand. Thus the bar is 20 cm long. It is to be noted that subdividing the same bar into eight 2.5-cm long cells gives the same final temperature after the same length of time.

To allow fairly accurate calculations without having to use many decimal places, we use an arbitrary but large amount of heat, 40,000 joules. This will raise cell 1's temperature by

$$\Delta T = \frac{Q}{mc} = \frac{40,000 \text{ J}}{1.12 \text{ kg} \cdot 383 \frac{\text{J}}{\text{kg} \cdot \text{K}}} \quad (3)$$

$$= 93.25^\circ\text{C} \text{ or } T = 0^\circ\text{C} + 93.25^\circ\text{C} = 93.25^\circ\text{C}.$$

We then install our insulating plug and watch the heat flow through the bar, proceeding in time intervals of one second through the iterative procedure shown in Table I.

Step (A) from Table I: Taking the internal energy of the bar (and of each cell) as zero before the application of the blowtorch and starting the timer immediately thereafter, at time $t = 0$ s the cell internal energies are

cell 1	cell 2	cell 3	cell 4
40,000 J	0 J	0 J	0 J

(Students should be aware that we assign zero joules to the last three cells because we wish to examine the effects of the heat added to cell 1, not because the cells contain no energy.)

Step (B) from Table I: The cell temperatures at $t = 0$ s are:

cell 1	cell 2	cell 3	cell 4
93.25°C	0°C	0°C	0°C

Step (C) from Table I: Now we look at how the heat flows from cell 1 to cell 2. The cross-sectional area A is $(0.05 \text{ m})^2$, and the distance between the centers of the cells is 0.05 m. The thermal conductivity k is 394 W/m·K; the time interval is 1 s, and the difference in temperature between the two cell centers is $93.25^\circ\text{C} - 0^\circ\text{C}$. Equation (2) tells us that in the first time interval we've transferred

$$Q = (93.25^\circ\text{C} - 0^\circ\text{C}) (0.05 \text{ m})^2 (394 \text{ W/m}\cdot\text{K}) (1 \text{ s}) / (0.05 \text{ m}) = 1837 \text{ J} \quad (4)$$

from cell 1 to cell 2 in the first second. We assume that there is no heat transfer between cell 2 and cell 3 because they are at equal temperatures, so there is no energy transfer across their common boundary; likewise between cells 3 and 4.

Return to Step (A): Accounting for the internal energy in each cell, we note that we have lost 1837 J from cell 1 to cell 2. Here is the way things now stand:

cell 1	cell 2	cell 3	cell 4
38,163 J	1837 J	0 J	0 J

Step (B): Now we calculate the new temperature in each cell. 1837 J of heat has flowed out of cell 1. It has a mass of 1.12 kg, and the specific heat of copper is 383 J/kg·K, so the temperature of cell 1 is now changed by Q/mc :

$$\begin{aligned} 93.25^\circ\text{C} + \Delta T &= 93.25^\circ\text{C} + \frac{Q}{mc} \\ &= 93.25^\circ\text{C} + \frac{-1837 \text{ J}}{(1.12 \text{ kg})(383 \frac{\text{J}}{\text{kg}\cdot\text{K}})} \\ &= 93.25^\circ\text{C} - 4.28^\circ\text{C} = 88.97^\circ\text{C}, \end{aligned} \quad (5)$$

and in cell 2 it is $1837 \text{ J} / [(1.12 \text{ kg}) (383 \text{ J/kg}\cdot\text{K})] = 4.28^\circ\text{C}$, so our new cell temperatures are

cell 1	cell 2	cell 3	cell 4
88.97°C	4.28°C	0°C	0°C

after one second.

During the second time interval, heat will continue to flow from cell 1 to cell 2 and now, because there is a temperature differential across their common boundary, from cell 2 to cell 3. Again we assume that since there was no temperature difference across the boundary between cells 3 and 4 at the start of the time interval, no heat flows into cell 4 during this interval.

Step (C): Using the procedures in the previous calculation, we find that during our second 1-s time interval 1668 J is transferred from cell 1 to cell 2, and 84 J flows from cell 2 to cell 3.

Return to Step (A): Now we must update our energy accounts (A). Clearly 1668 J has left cell 1 while cell 2 has gained 1668 J and lost 84 J to cell 3. Our new cell internal energies are thus

cell 1	cell 2	cell 3	cell 4
36,495 J	3421 J	84 J	0 J

Step (B): Our new cell temperatures are

cell 1	cell 2	cell 3	cell 4
85.08°C	7.97°C	0.20°C	0°C

Step (C): At the end of the third 1-s time interval, the following transfers will have occurred:

cell 1 to 2: 1386 J, **cell 2 to 3:** 209 J, **cell 3 to 4:** 11 J .

Return to Step (A): The new cell internal energies will thus be

cell 1	cell 2	cell 3	cell 4
35,109 J	4598 J	282 J	11 J

time	$T_1=T_{1_{prev}}-Q_{12}/mc$	$T_2=T_{2_{prev}}+(Q_{12}-Q_{23})/mc$	$T_3=T_{3_{prev}}+(Q_{23}-Q_{34})/mc$	$T_4=T_{4_{prev}}+Q_{34}/mc$	$Q_{12}=(kA/L)\Delta t(T_1-T_2)$	$Q_{23}=(kA/L)\Delta t(T_2-T_3)$	$Q_{34}=(kA/L)\Delta t(T_3-T_4)$
0	93.25	0	0	0	1837.00	0	0
1	88.97	4.28	0.00	0.00	1668.27	84.36	0.00

Fig. 2. The first three rows of an Excel spreadsheet programmed to solve Eqs. (1) and (2). The values in red were computed by hand and used as initial values for the program. The values in black were computed by the program.

Step (B): giving the new temperatures

cell 1 cell 2 cell 3 cell 4
 81.54°C 11.16°C 0.54°C 0.01°C

And so we watch the heat diffuse through the bar with each iteration.

Example Using Spreadsheet Over 300 Iterations

The simulation can be run for any amount of time and to any required accuracy in a spreadsheet such as Excel. In our example,

Column A is time.

Columns B–E are the temperatures of the four cells computed from the specific heat equation

$$T = T_{\text{previous}} + \Delta T = T_{\text{previous}} + \frac{Q_{\text{in}} - Q_{\text{out}}}{mc} \quad (6)$$

Columns F–H are the heats transferred between two adjacent cells, Q_{12} , Q_{23} , and Q_{34} , and are computed from the heat conduction equation

$$Q_{ij} = \frac{kA}{L} \Delta t (T_i - T_j) \quad (7)$$

Each new row increments the solution by Δt .

Cells A1–E1 contain the various physical constants needed for the simulation as described.

Figure 2 displays the first three rows of our Excel spreadsheet for the copper example presented in this paper. Figure 3 presents the solution after five minutes or 300 iterations.

Lessons From the Graphs

This graph illustrates a great deal of thermodynamic theory in a very compact form. First, note that the graphs have no minima; that is, the temperature of a cell never goes down and then back up. If there were a minimum of this sort, it would mean that the cell got cooler and then somehow regained its heat from its

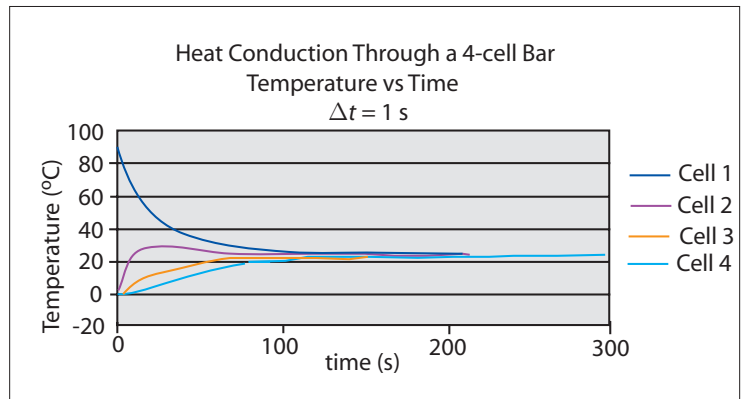


Fig. 3. The temperature of each cell as a function of time.

surroundings, which doesn't happen in a closed system.

Note also that the curves never dip beneath the time axis, thus reassuring us that a cell never becomes cooler than it was at the start of the experiment. And finally, note that all the cells ultimately warm up to equal temperatures, though it will take quite a while for this to happen.

References

1. $\nabla \cdot (k\nabla T) = \rho c \frac{\partial T}{\partial t}$. See for example H. McGee, J. McInerney, and A. Harrus, "The virtual cook: Modeling heat transfer in the kitchen," *Phys. Today* **52**, 30–36 (Nov. 1999).
2. For example, Harvey E. White, *Modern College Physics* (D. Van Nostrand Co., 1966).
3. *Marks' Standard Handbook for Mechanical Engineers*, edited by E.A. Avallone and T. Baumeister (McGraw-Hill, 1996) or www.copperinfo.com/aboutcopper/properties.html.

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